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Q Evaluation of Antennas in an Electrically Conductive Medium

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***Abstract*—The quality factor is analyzed for the spherical modes which represent fields of an antenna in an electrically lossy medium. Like the previous approach in free space, the stored energy in the evanescent field is represented by subtracting the energy associated with the radiation field from the total field energy. Instead of evaluating the stored energy over the whole space, simple equations on Q using the radiation efficiency and the medium property are derived using the concept of power dissipation in a lossy medium and the relation between Q of TM and TE modes. The theory is verified by the comparison of Q's evaluated from the impedance bandwidths of TM and TE modes and the proposed equations in a lossy medium.**

***Index Terms*—Antenna theory, lossy systems, quality factor, spherical mode, stored energy.**

I. INTRODUCTION

The quality factor or Q concept in an antenna has been used as a measure of bandwidth limitation by the definition and the relation

$$Q = \frac{\omega|W_{\rm int}|}{P} \approx \frac{1}{{\rm FBW}_{cd}} \approx \frac{2}{{\rm FBW}_{V}} \tag{1}$$

when $Q \gg 1$ ($Q > 2$ often suffices), where $W_{\rm int}$ is the total internal energy of the antenna tuned (to have zero input reactance), P is the input power, and ${\rm FBW}_{cd}$ and ${\rm FBW}_{V}$ are the fractional half-power bandwidth of conductance with a constant voltage source and of the matched voltage standing wave ratio (VSWR), respectively [1], [2]. In earlier works, Q of an antenna was evaluated by the combination of Q's of the spherical modes, which represent the radiated fields of the antenna [3], [4]. Because Q of the spherical mode is solved by considering only the region outside a sphere within which an antenna is assumed to be placed, it is a partial Q of an actual antenna and a true limit for an ideal antenna not having the internal energy inside the sphere. That is

$$Q_{\rm exact} = Q^{\rm in} + Q^{\rm out} \geq Q^{\rm out} \tag{2}$$

where $Q_{\rm exact}$ is the exact Q of the actual antenna, $Q^{\rm in}$ and $Q^{\rm out}$ are divided Q's by the surface of the sphere, and $Q^{\rm out}$ is obtainable from Q of the spherical mode. Therefore, Q of the spherical mode was used to estimate the minimum Q or the maximum bandwidth of an unspecified antenna with a given size [3], [4]. These works were for antennas in free space; however, nowadays there are increasing cases of antennas in a lossy medium. Especially, in the medical field, for wireless transmission of information from inside a human body, a small transceiver is implanted or swallowed into the human body, which is known as an electrically lossy medium [5]. Then, for the estimation of the impedance

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bandwidth and efficiency of the antenna in such systems, Q of spherical modes in an electrically lossy medium needs to be evaluated.

In the evaluation of Q of the spherical mode, we found two definitions of the stored or internal energy, W', in the region outside the sphere in free space. The first is given by [6] as

$$W' = W - W_r \tag{3}$$

where W and W_r are the energy associated with the total field and the radiation field, respectively. Although this is conceptually defined by intuition, its solution results in the equal energy stored in the reactive elements of the equivalent circuit of the spherical mode in [3]. The second is mentioned by [7] and confirmed in [1] as

$$W' = W - \lim_{r\to\infty} r\frac{P_r(r)}{c} \tag{4}$$

where $P_r(r)$ is the radiated power at distance r and c is the velocity of energy flow. This was rigorously derived from the frequency derivative of the input reactance of the antenna. Equations (3) and (4) are shown to be nearly equal in free space for an electrically small antenna [1, Appendix C]. But if we assume the space is filled with a medium infinitely, and there is even a very little loss in the medium, (4) becomes very large because the second term in (4) becomes zero, owing to the exponential attenuation of the radiated power. So, we used the first definition.

In this letter, Q of the spherical modes in an electrically lossy medium is analyzed in a similar manner as in the earlier works, and the relation between Q's of TM and TE modes is found with their radiation efficiency. By the concept of separating the radiated power and the dissipated power in a medium, Q of TM modes could be obtained by the radiation efficiency and the medium property, and using the relation between Q's of TM and TE modes, Q of TE modes is derived as a simple equation.

II. Q OF THE SPHERICAL MODES IN FREE SPACE

Generally the quality factor or Q of an antenna is defined as [8]

$$Q = \begin{cases} \frac{2\omega W_e'}{P_r} & W_e' > W_m' \\ \frac{2\omega W_m'}{P_r} & W_m' > W_e' \end{cases} \tag{5}$$

where W_e' and W_m' are the average stored electric and magnetic energy, and P_r is the radiated power. To evaluate (5) for an antenna, equivalent circuits of the spherical modes that represent fields of the antenna outside a sphere boundary were derived and the energy stored in the reactive elements of the circuit was considered as the stored energy of the antenna [3]. Later the stored energy was obtained from the energy density stored in the evanescent fields over the whole space by subtracting the energy associated with the radiation field [6]. In [6], considering the fields of an antenna outside a sphere of radius a, the stored energy in free space is given by

$$W_e' + W_m' = \int_a^\infty\int_0^{2\pi}\int_0^\pi (w_e + w_m) r^2 \sin\theta d\theta d\phi dr - \int_a^\infty \frac{P_r}{c} dr \tag{6}$$

where $w_e = (1/4)\epsilon|\mathbf{E}|^2$, $w_m = (1/4)\mu|\mathbf{H}|^2$, and $c = 1/\sqrt{\epsilon\mu}$ are the electric, magnetic field energy density, and the velocity of energy flow with $\epsilon = \epsilon_0$ and $\mu = \mu_0$. (6) can be rewritten as in [9]

$$W_e' = \int_a^\infty\int_0^{2\pi}\int_0^\pi (w_e - w_{e.r}) r^2 \sin\theta d\theta d\phi dr \tag{7}$$

$$W_m' = \int_a^\infty\int_0^{2\pi}\int_0^\pi (w_m - w_{m.r}) r^2 \sin\theta d\theta d\phi dr \tag{8}$$

where $w_{e.r} = (1/4)\epsilon|\mathbf{E}_{\rm rad}|^2$ and $w_{m.r} = (1/4)\mu|\mathbf{H}_{\rm rad}|^2$ are the electric and magnetic energy density associated with the radiation field.

The radiated power can be obtained from the complex power S that is given by the integration of the complex Poynting vector over the surface of r as

$$\begin{aligned} S(r) &= \frac{1}{2}\int_0^{2\pi}\int_0^\pi (\mathbf{E}\times\mathbf{H}^*)\cdot\hat{\mathbf{r}} r^2 \sin\theta d\theta d\phi \\ &= P_r + j2\omega\left\{W_m(r) - W_e(r)\right\} \end{aligned} \tag{9}$$

where the radiated power is independent of the surface of r assuming lossless in free space.

To evaluate Q of TM_{0n} spherical modes, let us set the magnetic vector potential as

$$A_r = \hat{H}_n(kr) P_n(\cos\theta)$$

where $\hat{H}_n(x) = x h_n(x)$ is other type of the spherical Hankel function of the second kind used in [8]. Then, the field components are given as

$$\begin{aligned} E_r &= \eta\frac{n(n+1)}{jkr^2}\hat{H}_n(kr) P_n(\cos\theta) \\ E_\theta &= \frac{\eta}{jr}\hat{H}_n'(kr) P_n^1(\cos\theta) \\ H_\phi &= -\frac{1}{r}\hat{H}_n(kr) P_n^1(\cos\theta) \end{aligned} \tag{10}$$

where $\hat{H}_n'(kr) = \partial\hat{H}_n(kr)/\partial(kr)$.

The complex power becomes

$$\begin{aligned} S &= \frac{1}{2}\int_0^{2\pi}\int_0^\pi E_\theta H_\phi^* r^2 \sin\theta d\theta d\phi \\ &= \frac{1}{2}\lambda_n j\eta \hat{H}_n'(kr)\hat{H}_n(kr)^* \end{aligned} \tag{11}$$

where $\lambda_n = \int_0^{2\pi}\int_0^\pi P_n^1(\cos\theta)^2 \sin\theta d\theta d\phi = (4\pi n(n+1))/(2n+1)$.

Using $\lim_{x\to\infty}\hat{H}_n(x) = j^{n+1}e^{-jx}$ in (11) at $r\to\infty$, the radiated power can be easily obtained by (9) as

$$P_r = \lim_{r\to\infty} \mathrm{Re}\left\{S(r)\right\} = \frac{1}{2}\lambda_n\eta \tag{12}$$

and using (10) in (7), the stored electric energy, which is larger than magnetic energy in TM modes, is given as

$$\begin{aligned} W_e' &= \frac{1}{4}\epsilon\int_a^\infty\int_0^{2\pi}\int_0^\pi \left(|E_r|^2 + |E_\theta|^2 - \left|E_\theta^{\rm rad}\right|^2\right) r^2 \sin\theta d\theta d\phi dr \\ &= \frac{1}{4}\lambda_n\mu\int_a^\infty \left[\frac{n(n+1)}{(kr)^2}\left|\hat{H}_n(kr)\right|^2 + \left|\hat{H}_n'(kr)\right|^2 - 1\right] dr \end{aligned} \tag{13}$$

where $\int_0^{2\pi}\int_0^{\pi} P_n(\cos\theta)^2 \sin\theta d\theta d\phi = 4\pi/(2n+1)$ and $E_\theta^{\mathrm{rad}} = -(\eta/r)e^{-jkr}P_n^1(\cos\theta)$.

By the integration of (13) and using (12), Q can be given as

$$\begin{aligned} Q_n &= \frac{2\omega W_e'}{P_r} \\ &= ka - \frac{ka}{2}\left|\hat{H}_n(ka)\right|^2 + \frac{ka}{2} \\ &\quad \times \left\{\hat{J}_{n-1}(ka)\hat{J}_{n+1}(ka) + \hat{N}_{n-1}(ka)\hat{N}_{n+1}(ka)\right\} \\ &\quad - \hat{J}_n(ka)\hat{J}_n'(ka) - \hat{N}_n(ka)\hat{N}_n'(ka) \end{aligned} \tag{14}$$

where $\hat{J}_n(x) = xj_n(x)$ and $\hat{N}_n(x) = xy_n(x)$ are other types of the spherical Bessel functions of the first and second kinds, with $\hat{H}_n(x) = \hat{J}_n(x) - j\hat{N}_n(x)$. This is the same result in [6] with a different notation.

For Q of TE_{0n} modes, if we set the electric vector potential as

$$F_r = \eta\hat{H}_n(kr)P_n(\cos\theta).$$

Then the field components are given as

$$\begin{aligned} H_r &= \frac{n(n+1)}{jkr^2}\hat{H}_n(kr)P_n(\cos\theta) \\ H_\theta &= \frac{1}{jr}\hat{H}_n'(kr)P_n^1(\cos\theta) \\ E_\phi &= \frac{\eta}{r}\hat{H}_n(kr)P_n^1(\cos\theta). \end{aligned} \tag{15}$$

The complex power becomes

$$\begin{aligned} S &= -\frac{1}{2}\int_0^{2\pi}\int_0^{\pi} E_\phi H_\theta^* r^2 \sin\theta d\theta d\phi \\ &= -\frac{1}{2}\lambda_n j\eta\hat{H}_n(kr)\hat{H}_n'(kr)^* \end{aligned} \tag{16}$$

The radiated power is same as in (12), and using (15) in (8), the stored magnetic energy becomes equal to (13). Therefore, Q of TE_{0n} modes is same as (14) in free space.

III. Radiation Efficiency in a Lossy Medium

Assuming an antenna in a lossy medium, the real part of the complex power is not only the radiated power but also the dissipated power in the evanescent fields, P_d, as

$$S(r) = P_r + P_d + j2\omega\{W_m - W_e\}. \tag{17}$$

In the far-field region, only the radiated power exists and attenuates from the surface of radius a as

$$\begin{aligned} P_r(r) &= \mathrm{Re}\{S(r)\}, \quad as \quad r \to \infty \\ &= P_r(a)e^{-2|\mathrm{Im}(k)|(r-a)}. \end{aligned} \tag{18}$$

The radiated power at a distance r can be expressed in terms of the input power into the surface of radius a, the radiation efficiency, η_{eff}, and the attenuation factor as in [10]

$$P_r(r) = \eta_{\mathrm{eff}}\mathrm{Re}\{S(a)\}e^{-2|\mathrm{Im}(k)|r} \tag{19}$$

$$where \quad \eta_{\mathrm{eff}} = \frac{P_r(a)}{\mathrm{Re}\{S(a)\}}e^{2|\mathrm{Im}(k)|a}. \tag{20}$$

The radiation efficiency of TM_{0n} and TE_{0n} modes in a lossy medium can be evaluated by using (11) and (16) in (18) and the results give the same radiated power for both modes as

$$\begin{aligned} P_r(r) &= \lim_{r\to\infty}\mathrm{Re}\{S(r)\} \\ &= \frac{1}{2}\lambda_n\mathrm{Re}(\eta)e^{-2|\mathrm{Im}(k)|r} \end{aligned}$$

then by (19) the radiation efficiency is given as

$$\eta_{\mathrm{eff}} = \begin{cases} \frac{\mathrm{Re}\{\eta\}}{\mathrm{Re}\left\{j\eta\hat{H}_n'(ka)\hat{H}_n(ka)^*\right\}} & \text{TM mode} \\ \frac{\mathrm{Re}\{\eta\}}{\mathrm{Re}\left\{-j\eta\hat{H}_n(ka)\hat{H}_n'(ka)^*\right\}} & \text{TE mode} \end{cases}. \tag{21}$$

This is the same result in [10] with a different notation.

IV. Q of the Spherical Modes in an Electrically Lossy Medium

A. Q's of TM and TE Modes and Their Relationship

Q of spherical modes in a lossy medium will be considered on the assumption that the medium is homogeneous, infinite, and only electrically conductive. Also, its dispersion is assumed to be nearly negligible in the interested bandwidth, that is, $|\partial\epsilon_r/\partial\omega| \ll \epsilon_r/\omega$ and $|\partial\sigma/\partial\omega| \ll \sigma/\omega$, where $\epsilon = \epsilon_r - j\epsilon_i$, $\epsilon_i = \sigma/\omega$, and $\mu = \mu_0$. For the evaluation of Q, the energy associated with the radiation fields in a lossy medium should be subtracted like in free space problem; otherwise, Q becomes very large when the loss is small. Therefore, Q of TM modes, by using $w_e = (1/4)\epsilon_r|\mathbf{E}|^2$ as the electric energy density, is given by

$$\begin{aligned} Q_{\mathrm{TM}} &= \frac{2\omega W_e'}{P} = \frac{2\omega W_e'}{\mathrm{Re}\{S(a)\}} \\ &= \omega\mu\frac{\epsilon_r}{|\epsilon|}\int_a^\infty \left\{\frac{n(n+1)}{(kr)^2}\left|\hat{H}_n(kr)\right|^2 \right. \\ &\qquad \left. + \left|\hat{H}_n'(kr)\right|^2 - e^{-2|\mathrm{Im}(k)|r}\right\}dr \\ &\quad /\mathrm{Re}\left\{j\eta\hat{H}_n'(ka)\hat{H}_n(ka)^*\right\}. \end{aligned} \tag{22}$$

Q of TE modes by (8) and (16) is given as

$$\begin{aligned} Q_{\mathrm{TE}} &= \frac{2\omega W_m'}{P} = \frac{2\omega W_m'}{\mathrm{Re}\{S(a)\}} \\ &= \omega\mu\int_a^\infty \left\{\frac{n(n+1)}{(kr)^2}\left|\hat{H}_n(kr)\right|^2 \right. \\ &\qquad \left. + \left|\hat{H}_n'(kr)\right|^2 - e^{-2|\mathrm{Im}(k)|r}\right\}dr \\ &\quad /\mathrm{Re}\left\{-j\eta\hat{H}_n(ka)\hat{H}_n'(ka)^*\right\}. \end{aligned} \tag{23}$$

Comparing (22) and (23) with (21), a relation between $Q_{\rm TM}$ and $Q_{\rm TE}$ can be found in terms of the radiation efficiency as

$$\cos\delta \frac{Q_{\rm TE}}{\eta_{\rm eff.TE}} = \frac{Q_{\rm TM}}{\eta_{\rm eff.TM}} \tag{24}$$

where $\delta = \tan^{-1}(\epsilon_i/\epsilon_r)$.

B. Q Derivation in Terms of Radiation Efficiency and Medium Property

If we consider Q from the electric energy stored in the electric field and the supplied power which is dissipated by the electric field in a volume, the Q, denoted as Q_d, can be given only by the medium property as

$$\begin{aligned} Q_d &= \frac{2\omega W_e}{P} = 2\omega \frac{\int_V \frac{1}{4}\epsilon_r |\mathbf{E}|^2 dV}{\int_V \frac{1}{2}\sigma |\mathbf{E}|^2 dV} \\ &= \frac{\epsilon_r}{\epsilon_i} = \frac{1}{\tan\delta}. \end{aligned}$$

Then Q of TM modes can be expressed in terms of Q_d and the radiation efficiency by (20) as

$$\begin{aligned} Q_{\rm TM} &= \frac{2\omega W_e'}{P} = \frac{2\omega W_e}{P} - \frac{2\omega W_{e.r}}{P} \\ &= \frac{2\omega W_e}{P} - \frac{P_r}{\mathrm{Re}\,\{S(a)\}} \frac{2\omega W_{e.r}}{P_r} \\ &= \left(1 - \eta_{\rm eff.TM} e^{-2|\mathrm{Im}(k)|a}\right) \cot\delta \end{aligned} \tag{25}$$

where $W_{e.r}$ is the electric energy associated with the radiation field and $2\omega W_e/P = 2\omega W_{e.r}/P_r = Q_d$. Equation (25) converges to (14) when $\delta \to 0$ as expected and proved in the Appendix. Using (25) in (24), Q of TE modes can be obtained without the evaluation of (23) as

$$Q_{\rm TE} = \eta_{\rm eff.TE} \left(\frac{1}{\eta_{\rm eff.TM}} - e^{-2|\mathrm{Im}(k)|a} \right) \csc\delta. \tag{26}$$

V. Numerical Examples

For the verification of (25) and (26), Q from the half-power bandwidth of the reflection coefficient is compared as in [2]

$$Q \approx Q_{3\ \rm dB} = \frac{2\omega_0}{\rm Half - Power\ Bandwidth} \tag{27}$$

when $Q \gg 1$ ($Q \gtrsim 2$ often suffices). The reflection coefficients are given by using the impedance of TM and TE modes matched by series inductor and parallel capacitance, respectively, at ω_0 as

$$\Gamma_{\rm TM}(\omega) = \frac{Z_{\rm TM}(\omega) + j\omega L - \mathrm{Re}\,\{Z_{\rm TM}(\omega)\}}{Z_{\rm TM}(\omega) + j\omega L + \mathrm{Re}\,\{Z_{\rm TM}(\omega)\}}$$

$$\text{where} \quad L = \frac{\mathrm{Im}\,\{Z_{\rm TM}(\omega_0)\}}{\omega_0}, \quad Z_{\rm TM} = \frac{E_\theta}{H_\phi} = j\eta \frac{\hat{H}_n'(ka)}{\hat{H}_n(ka)}$$

$$\Gamma_{\rm TE}(\omega) = -\frac{Y_{\rm TE}(\omega) + j\omega C - \mathrm{Re}\,\{Y_{\rm TE}(\omega)\}}{Y_{\rm TE}(\omega) + j\omega C + \mathrm{Re}\,\{Y_{\rm TE}(\omega)\}}$$

$$\text{where} \quad C = \frac{\mathrm{Im}\,\{Y_{\rm TE}(\omega_0)\}}{\omega_0}, \quad Y_{\rm TE} = -\frac{H_\theta}{E_\phi} = \frac{j}{\eta} \frac{\hat{H}_n'(ka)}{\hat{H}_n(ka)}.$$

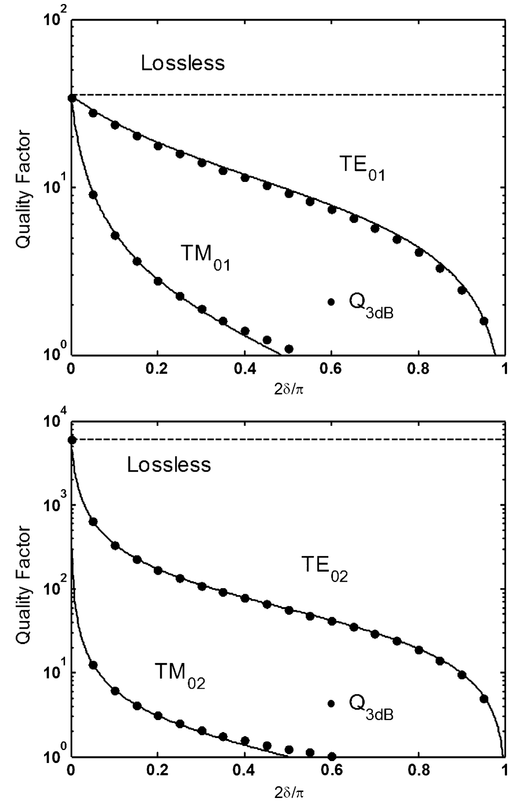

Fig. 1. Q of TM_{01}, TE_{01}, TM_{02} and TE_{02} modes with $a = 0.05\lambda$ and $\epsilon_r = \mu = \omega = 1$.

Fig. 1 shows Q of TM_{01}, TE_{01}, TM_{02}, and TE_{02} modes with $a = 0.05\lambda$ and $\epsilon_r = \mu = \omega = 1$. In Fig. 1, Q's of (25) and (26) agree well with (27) and converge to (14) when $\delta \to 0$. One can see that the magnetic type antenna, i.e., loop, which generates TE modes has higher Q than the electric type antenna, i.e., dipole, which generates TM modes at the cost of its superior radiation efficiency in the electrically conducting media as shown in [10].

VI. Conclusion

Q of spherical modes in an electrically conductive medium is derived as simple equations in terms of radiation efficiency and medium property. The theory was verified by the comparison of Q's evaluated from the impedance bandwidths of TM and TE modes and the proposed equations in a lossy medium. This result will be helpful to estimate easily the lower bound on Q, the bandwidth, and the efficiency of an antenna emerging in an electrically lossy medium.

Appendix

Proof that $Q_{\rm TM}$ of (25) converges to Q_n of (14) as $\delta \to 0$: The denominator of $\eta_{\rm eff.TM}$ in (21) is rearranged as

$$\begin{aligned} \mathrm{Re}\left\{ j\eta \hat{H}_n'(ka)\hat{H}_n(ka)^* \right\} &= \mathrm{Re}\,\{\eta(A + jB)\} \\ &= \mathrm{Re}\{\eta A\} - \mathrm{Re}\{\eta\}\mathrm{Im}\{B\} - \mathrm{Im}\{\eta\}\mathrm{Re}\{B\} \end{aligned} \tag{28}$$

where

$$A = \hat{J}_n(ka)^* \hat{N}_n'(ka) - \hat{N}_n(ka)^* \hat{J}_n'(ka)$$
$$B = \hat{J}_n'(ka)\hat{J}_n(ka)^* + \hat{N}_n'(ka)\hat{N}_n(ka)^*.$$

For brevity, the loss tangent term is denoted as Δ

$$\Delta = \tan\delta.$$

Then $\delta \to 0$ means $\Delta \to 0$.

As $\Delta \to 0$, by the Wronskian of Bessel's equation, A becomes

$$A = \hat{J}_n(ka)^* \hat{N}_n'(ka) - \hat{N}_n(ka)^* \hat{J}_n'(ka) \approx 1 \tag{29}$$

and the wave impedance and the wave constant are approximated as

$$\eta \approx \eta_0\left(1 + j\frac{\Delta}{2}\right), \quad k \approx k_0\left(1 - j\frac{\Delta}{2}\right) \tag{30}$$

where η_0 and k_0 are the values of lossless medium with $\delta = 0$, and using (30), the following terms are approximated as

$$\hat{J}_n(ka) \approx \hat{J}_n(k_0a) - j\frac{\Delta}{2}k_0a\hat{J}_n'(k_0a) \tag{31}$$
$$\hat{J}_n'(ka) \approx \hat{J}_n'(k_0a) - j\frac{\Delta}{2}k_0a\hat{J}_n''(k_0a). \tag{32}$$

Using (31) and (32), and neglecting the term multiplied by Δ^2, the first term in B is approximated as

$$\hat{J}_n'(ka)\hat{J}_n(ka)^* \approx \hat{J}_n(k_0a)\hat{J}_n'(k_0a) + j\frac{\Delta}{2}k_0a\left\{\hat{J}_n'(k_0a)^2 - \hat{J}_n(k_0a)\hat{J}_n''(k_0a)\right\} \tag{33}$$

and, in the same way, the second term in B is approximated in the same form.

Using (29), (30), and (33) in (28) results in

$$\begin{aligned}
&\mathrm{Re}\{\eta A\} - \mathrm{Re}\{\eta\}\mathrm{Im}\{B\} - \mathrm{Im}\{\eta\}\mathrm{Re}\{B\} \\
&\approx \eta_0 + \eta_0\Delta\Bigg[k_0a - \frac{k_0a}{2}\Big\{\hat{J}_n'(k_0a)^2 + \hat{N}_n'(k_0a)^2 \\
&\qquad - \hat{J}_n(k_0a)\hat{J}_n''(k_0a) - \hat{N}_n(k_0a)\hat{N}_n''(k_0a)\Big\} \\
&\qquad - \frac{1}{2}\left\{\hat{J}_n(k_0a)\hat{J}_n'(k_0a) + \hat{N}_n(k_0a)\hat{N}_n'(k_0a)\right\}\Bigg].
\end{aligned} \tag{34}$$

The term within square brackets in (34) is equal to (14) by the following relationships of Bessel function:

$$\hat{J}_n'^2 = \frac{1}{x}\hat{J}_n\hat{J}_n' - \hat{J}_{n-1}\hat{J}_{n+1} + \frac{n(n+1)}{x^2}\hat{J}_n^2$$
$$\hat{J}_n'' = \left\{\frac{n(n+1)}{x^2} - 1\right\}\hat{J}_n.$$

Then (28) is approximated as

$$\mathrm{Re}\left\{j\eta\hat{H}_n'(ka)\hat{H}_n(ka)^*\right\} \approx \eta_0 + \eta_0\Delta Q_n$$

and Q_{TM} of (25) converges to

$$\begin{aligned}
Q_{\mathrm{TM}} &= \left[1 - \frac{\mathrm{Re}\{\eta\}e^{-2|\mathrm{Im}(k)|a}}{\mathrm{Re}\left\{j\eta\hat{H}_n'(ka)\hat{H}_n(ka)^*\right\}}\right]\cot\delta \\
&\approx \left(1 - \frac{\eta_0}{\eta_0 + \eta_0\Delta Q_n}\right)\frac{1}{\Delta} \\
&= Q_d \| Q_n \\
&\approx Q_n
\end{aligned}$$

as $\Delta \to 0$, where $Q_d \| Q_n = \left(Q_d^{-1} + Q_n^{-1}\right)^{-1}$.

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